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# A COMPLEMENTARITY ALGORITHM FOR OPTIMAL STATIONARY PROGRAMS IN GROWTH MODELS WITH QUADRATIC UTILITY

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and

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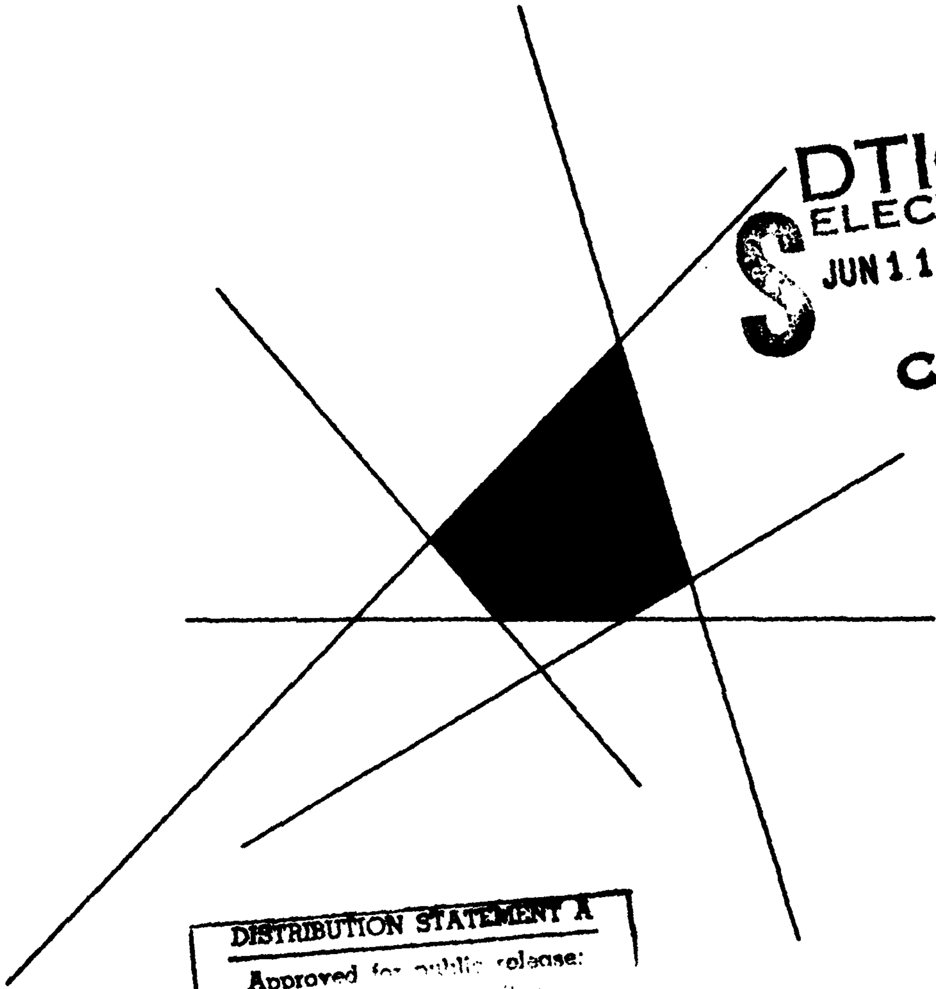





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PROGRAMS IN GROWTH MODELS WITH QUADRATIC UTILITY

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## ABSTRACT

It is shown that under natural economic assumptions the complementary pivot algorithm will find an optimal steady state for a von Neumann growth model with a quadratic utility function. This generalizes the result of Dantzig and Mann for the case of linear utility.

# A COMPLEMENTARITY ALGORITHM FOR OPTIMAL STATIONARY PROGRAMS IN GROWTH MODELS WITH QUADRATIC UTILITY

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## 1. INTRODUCTION

Some years ago Sutherland [5] and Koopmans [3] independently proved the existence of optimal stationary programs for growth models in which future utility is discounted at a constant rate. They also showed that under suitable assumptions such programs could be characterized as ones which satisfy a simple one period "competitive condition." Subsequently, Hansen and Koopmans [2] gave another proof of this characterization and showed how techniques for calculating approximate fixed points of continuous mappings could be used to approximate these stationary solutions. An alternative approach was then proposed by Dantzig and Mann [1] who showed that for the special case of a linear utility function one could obtain the optimal stationary program by applying Lemke's complementary pivot algorithm. They then proposed approximating an arbitrary concave utility by piecewise linear utilities to handle the general case.

The main purpose of this note is to show that the result of [1] remains valid when the linear utility function is generalized to any concave quadratic function. This represents an improvement from the economic point of view since, as has often been observed, linear functions are unsatisfactory for representing utilities since they imply, at least for Leontief technologies, that only one good will be consumed. Quadratic functions, on the other hand, may be thought of as a good first approximation to a general concave utility function.

A secondary objective is to reformulate the "key hypothesis" of [1] in a form in which its economic meaning becomes more evident. Finally, our proof has the merit of brevity. We assume, however, familiarity with the complementary pivot algorithm whose essential properties will be given in Section 3.

## 2. THE MODEL AND THE COMPETITIVE CONDITIONS

We are given $m$ by $n$ matrices $A$ and $B$ , $B \geqq 0$ , an m-vector $b$ , a concave differentiable (utility) function $\omega$ defined on $R^n$ and a discount factor $\delta$ on $(0,1)$ .

A *program* is a sequence $(x_t)$ of non-negative n-vectors such that

$$(2.1) \qquad Ax_{t+1} \lesssim Bx_t + b \qquad t = 0,1, \ldots .$$

The *value* $v$ of the program $(x_t)$ is given by

$$v((x_t)) = \sum_{t=0}^{\infty} \delta^t \omega(x_t) .$$

A program is *stationary* if $x_t = x$ for all $t$ and is *optimal stationary* if

$$v((x)) \gtrsim v((x_t))$$

for all programs $x_t$ with $x_o = x$ .

<u>Definition</u>:

The stationary program $(x)$ is *competitive* if there is a non-negative (price) m-vector $p$ such that

$$(2.2) \qquad (A - B)x \leqq b$$

and

$$(2.2)' \qquad p(A - B)x = pb$$

(2.3) $$p(A - \delta B) \geqq \nabla\omega(x)$$

and

(2.3)' $$p(A - \delta B)x = x\nabla\omega(x) \ .$$

The term competitive reflects the fact that the conditions above are satisfied precisely when $x$ maximizes "profit", defined as $\omega(x) + \delta pBx - pAx$, among all $x$ satisfying (2.2).

The following basic result is also proved in [5] and [2]. For the sake of completeness we reprove it here.

Theorem 1:

A competitive program is optimal.

Let $(x_t)$ be any program with $x_o = x$ where $(x)$ is competitive. From (2.3) we have

$$x_t\nabla\omega(x) \leq p(A - \delta B)x_t$$

and multiplying by $\delta^t$ and summing from 0 to T gives

$$\sum_{t=0}^{T} \delta^t x_t \nabla\omega(x) \leq p \sum_{t=0}^{T} \delta^t (A - \delta B)x_t$$

$$= p\left(Ax_o + \sum_{t=1}^{T} \delta^t (Ax_t - Bx_{t-1}) - \delta^{T+1}Bx_T\right)$$

$$\leqq p\left(Ax_o + \left(\sum_{t=1}^{T} \delta^t\right)b\right) \quad \text{(from (2.2) and } B \geqq 0)$$

$$= pAx + p(A - B)x \sum_{t=1}^{T} \delta^t \quad \text{(from (2.2)' and } x_o = x)$$

$$= p(A - \delta B)x \sum_{t=0}^{T} \delta^t + \delta^{T+1}pBx$$

$$= \left(\sum_{t=0}^{T} \delta^t\right)x\nabla\omega(x) + \delta^{T+1}pBx \quad \text{(from (2.3)')}$$

so

$$\sum_{t=0}^{T} \delta^t (x_t - x)\nabla\omega(x) \leq \delta^{T+1} pBx$$

and since $\delta^{T+1}pBx$ approaches zero we have

$$\sum_{t=0}^{\infty} \delta^t (x_t - x)\nabla\omega(x) \leqq 0 \ .$$

By concavity $\omega(x_t) - \omega(x) \leqq \nabla\omega(x)(x_t - x)$ so

$$\sum_{t=0}^{\infty} \delta^t \omega(x_t) \leq \sum_{t=0}^{\infty} \delta^t \omega(x) \quad \text{or} \quad v(x) \geqq v(x_t)$$

so $(x)$ is optimal ∎

The converse of Theorem 1 does not hold as shown by the following simple example:

$$A = 1 \ , \ B = 3 \ , \ \omega(x) = x \ , \ b = 0 \ , \ \delta = \frac{1}{2} \ .$$

The trivial program $x = 0$ is clearly optimal since it is the only program starting from $x_o = 0$ , but it is not competitive since $A - \delta B = -\frac{1}{2}$ , so (2.3) has no solution.

To get around this type of problem, Peleg and Ryder [4] have defined a technology to be *δ-productive if it contains* an input-output pair $(x,y)$ such that $x < \delta y$ . This is a rather natural condition which asserts that the model is capable of expanding at a rate greater than the reciprocal of the discount rate. For our purposes the weaker assumption $x \leqq \delta y$ will suffice. Intuitively, if this condition were not satisfied it would mean that the growth potential of the technology was not sufficient to compensate

for consumers' preference for present over future consumption. Thus, capital would be consumed faster than it could grow and the only possible steady state would be at the subsistence level. (For $b \geqq 0$ the only solution would be the trivial steady state $x = 0$.)

In our model the $\delta$-productivity condition becomes

$$(2.4) \qquad Ax - b \leqq \delta Bx \quad \text{for some} \quad x \geq 0 \; .$$

A second assumption we will make is that the model is not capable of achieving steady states with arbitrarily high utility. This is certainly reasonable for any economy in which production depends on exogenous bounded resources. Formally we have

(2.5) There exists $M > 0$ such that $\omega(x) \leq M$ for all $x$ satisfying (2.2).

Condition (2.4) and (2.5) will be used here instead of the "key hypothesis" used in [1].

## 3. THE CASE OF QUADRATIC UTILITY

We now consider the special case where $\omega$ is concave quadratic, that is,

$$\omega(x) = cx - \frac{1}{2} xDx$$

where $D$ is symmetric and positive semi-definite. In this case $\nabla\omega(x) = c - Dx$ . Substituting this in (2.3) and (2.3)' and introducing slack variables $u = b - (A - B)x$ , $v = p(A - \delta B) + Dx - c$ , we get the competitive condition in the form of the following Linear Complementarity Problem:

Find non-negative $(m+n)$-vectors $z = (x,p)$ and $w = (u,v)$ such that

$$w = Mz + q \ , \ wz = 0 \tag{3.1}$$

where $M = \begin{pmatrix} D & (A-\delta B)^T \\ B-A & 0 \end{pmatrix}$, $q = \begin{pmatrix} -c \\ b \end{pmatrix}$ .

Our main result is

**Theorem:**

If (2.4) and (2.5) are satisfied, then (3.1) has a solution.

Recall that the complementary pivot algorithm introduces the auxiliary vector $e$ all of whose coordinates are one, and either finds a solution of (3.1) or else non-negative solutions $(w,z,\theta)$ and $(\tilde{w},\tilde{z},\tilde{\theta})$ of

$$w - Mz - \theta e = q \ , \ wz = 0 \ , \ \theta > 0 \tag{3.2}$$

$$\tilde{w} - M\tilde{z} - \tilde{\theta} e = 0 \ , \ \tilde{w}\tilde{z} = 0 \ , \ \tilde{z} \neq 0 \tag{3.3}$$

$$\tilde{w}z = w\tilde{z} = 0 \ . \tag{3.4}$$

We must show that under our assumption this second alternative cannot occur. The key fact we need is that the matrix $M$ is "co-positive", that is,

Lemma:

If $z \geqq 0$ , then $zMz \geqq 0$ .

■ $M + M^T = \begin{pmatrix} 2D & (1-\delta)B^T \\ (1-\delta)B & 0 \end{pmatrix}$ and $(1 - \delta)B$ is non-negative.

Hence if $z = (x,p)$

$$2(zMz) = z(M + M^T)z = 2(xDx + p(1 - \delta)Bx) \geqq 0$$

since $D$ is positive semi-definite ■

Now suppose (3.2) - (3.4) hold. Multiplying (3.3) by $\tilde{z}$ gives

$$\tilde{z}M\tilde{z} + \tilde{\theta}(e\tilde{z}) = 0$$

so from the lemma, we get $\tilde{\theta} = 0$ and $\tilde{z}M\tilde{z} = 0$ since $\tilde{z} \neq 0$ . Letting $\tilde{z} = (\tilde{x},\tilde{p})$ this implies $\tilde{x}D\tilde{x} = 0$ which means

$$\tilde{x}D = 0 \tag{3.5}$$

by the property of positive semi-definite matrices. Since $\tilde{\theta} = 0$ , (3.3) becomes

$$\tilde{w} = M\tilde{z} \tag{3.6}$$

so multiplying by $z$ and using (3.4) we get

$$\tilde{z}M^T z = 0 \ . \tag{3.7}$$

Multiplying (3.2) by $\tilde{z}$ gives

$$(3.8) \qquad -\tilde{z}Mz - \theta e\tilde{z} = q\tilde{z}$$

and adding (3.7) and (3.8) gives

$$(3.9) \qquad -\tilde{z}(M + M^T)z - \theta(e\tilde{z}) = q\tilde{z} \ .$$

The second term on the left is negative since $\theta > 0$ . Now $\tilde{z}(M + M^T) = (2\tilde{x}D + \tilde{p}(1-\delta)B, (1-\delta)B\tilde{x}) = (1-\delta)(\tilde{p}B, B\tilde{x}) \geqq 0$ from (3.5). Hence the first term of (3.9) is non-positive so we have $q\tilde{z} < 0$ or

$$(3.10) \qquad -c\tilde{x} + p\tilde{b} < 0 \ .$$

Now from (2.4) we have

$$(3.11) \qquad (A - \delta B)x \leq b \quad \text{for some} \quad x \geqq 0$$

and from (3.6) $\tilde{p}(A - \delta B) \geqq 0$ so multiplying (3.11) by $\tilde{p}$ gives

$$0 \leqq \tilde{p}(A - \delta B)x \leq \tilde{p}b \ .$$

Therefore, from (3.10) we have $c\tilde{x} > 0$ but from (3.6), $(B - A)\tilde{x} \geqq 0$ . Now let $x$ be any solution of (2.2). Then $x + \lambda\tilde{x}$ , $\lambda > 0$ is also a solution and

$$\begin{aligned} \omega(x + \lambda\tilde{x}) &= \omega(x) + \lambda c\tilde{x} - \lambda\tilde{x}Dx - \lambda^2\tilde{x}D\tilde{x} \\ &= \omega(x) + \lambda c\tilde{x} \quad \text{from (3.5)} \end{aligned}$$

which is unbounded in $\lambda$ , contradicting (2.5) ■

## 4. AN EXAMPLE

Even when a competitive program exists, the complementary pivot algorithm may fail to find it as the following example shows:

$$A = \begin{pmatrix}1\\1\end{pmatrix} \quad B = \begin{pmatrix}2\\0\end{pmatrix} \quad b = \begin{pmatrix}-2\\3\end{pmatrix} \quad c = 1 \quad \delta < \frac{1}{2} .$$

The pair $x = 3$ , $p = (0,1)$ is competitive because

$$(A - B)x = \begin{pmatrix}-3\\3\end{pmatrix} \leqq \begin{pmatrix}-2\\3\end{pmatrix} , \; p(A - \delta B) = (0,1)\begin{pmatrix}1 - 2\delta\\1\end{pmatrix} = 1 = c \quad \text{and}$$

complementary slackness holds.

The initial tableau of the complementary pivot algorithm is

$$\begin{array}{cccccccc} e_1 & e_2 & e_3 & m_1 & m_2 & m_3 & e & q \end{array}$$

$$\left(\begin{array}{ccc|ccc|c|c} 1 & 0 & 0 & 0 & 2\delta - 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 3 \end{array}\right)$$

The first pivot gives

$$\begin{array}{c} e_1 \\ e \\ e_3 \end{array} \left(\begin{array}{ccc|ccc|c|c} 1 & -1 & 0 & 1 & 2\delta - 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 0 & 0 & 0 & 5 \end{array}\right)$$

and since $e_2$ leaves the basis, column $a^2$ must enter, but for $\delta \leq \frac{1}{2}$ , there is no feasible pivot and the algorithm terminates without finding the solution. Of course, in this case our condition (2.5) is not satisfied. For $\delta > \frac{1}{2}$ the algorithm, as proved, will find the solution. Letting $\varepsilon = 2\delta - 1$ , the tableau sequence is

$$\begin{array}{c} m_2 \\ e \\ e_3 \end{array} \left(\begin{array}{ccccccc|c} \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} & 0 & \frac{1}{\varepsilon} & 1 & -\frac{1}{\varepsilon} & 0 & \frac{1}{\varepsilon} \\ 0 & -1 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 0 & 0 & 0 & 5 \end{array}\right)$$

$$\begin{array}{c} m_1 \\ e \\ e_3 \end{array} \left(\begin{array}{ccccccc|c} 1 & -1 & 0 & 1 & \varepsilon & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -\varepsilon & 1 & 1 & 1 \\ -2 & 1 & 1 & 0 & -2\varepsilon & 2 & 0 & 3 \end{array}\right)$$

$$\begin{array}{c} m_1 \\ e \\ e_2 \end{array} \left(\begin{array}{ccccccc|c} -1 & 0 & 1 & 1 & -\varepsilon & 1 & 0 & 4 \\ -1 & 0 & 0 & 0 & -\varepsilon & 1 & 1 & 1 \\ -2 & 1 & 1 & 0 & -2\varepsilon & 2 & 0 & 3 \end{array}\right)$$

$$\begin{array}{c} m_1 \\ m_3 \\ e_2 \end{array} \left(\begin{array}{ccccccc|c} 0 & 0 & 1 & 1 & 0 & 0 & -1 & 3 \\ -1 & 0 & 0 & 0 & -\varepsilon & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 1 \end{array}\right)$$

Conclusion:

Under economically reasonable conditions the complementary pivot algorithm will always find an optimal stationary program.

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